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The structure of the Newtonian limit

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Abstract

We consider a smooth one-parameter family of four-dimensional manifolds X_ε, $\varepsilon \geq 0$, each one endowed with a covariant metric g_ε. It is assumed that g_ε is a Lorentz metric for each $\varepsilon > 0$, i.e., the signature of g_ε is $(+, -, -, -)$ for $\varepsilon > 0$, while the limit metric g_0 on X_0 is assumed to be degenerated of rank 1, i.e., the signature of g_0 is $(+, 0, 0, 0)$. We characterize when the limit manifold X_0 inherits the geometric structure of a Newtonian gravitation. The limit manifold X_0 is a Newtonian gravitation if and only if there exist the limits of the Levi-Civita connection ∇_ε, the curvature operator $\mathcal{R}_\varepsilon$ and the contravariant Einstein tensor G^2_ε as $\varepsilon \to 0$. Moreover, the existence of these limits is characterized in terms of the Taylor expansion of the family $\{g_\varepsilon\}$ with respect to the parameter ε.

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Keywords: Newtonian limit; Lorentz metric

1. Introduction

Works of Cartan [2,3] and Trautman [18] show that the Newtonian gravitation may be geometrically formulated as a four-dimensional manifold endowed with a covariant metric $\bar{g}$ of rank 1 (the time metric), a contravariant metric $\bar{g}^*$ of rank 3 (the space metric), a symmetric linear connection $\bar{\nabla}$ and a contravariant metric $\bar{T}^2$ (the matter tensor) satisfying certain conditions. This geometric formulation is the starting point of the rigorous study of the relations between Newtonian gravitation and general relativity, because any Lorentz metric g also defines a contravariant metric g^* (the dual metric), a symmetric linear connection ∇ (the Levi-Civita one) and a matter tensor (via the Einstein equation). Hence, it is full-sense

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to ask whether a given Newtonian gravitation is a deformation of certain Lorentz metrics or whether the limit of a sequence of Lorentz metrics is a Newtonian gravitation.

This paper is devoted to the study of the conditions that a family of Lorentz metrics must satisfy in order to have a Newtonian limit. Inspired in Ehlers' frame theory [4–6] and Rendall's paper [16], we shall consider a smooth one-parameter family of four-dimensional manifolds X_ε, $\varepsilon \geq 0$, each one endowed with a covariant metric g_ε. It is assumed that g_ε is a Lorentz metric for each $\varepsilon > 0$, i.e., the signature of g_ε is $(+,-,-,-)$ for $\varepsilon > 0$, while the limit metric g_0 on X_0 is assumed to be degenerated of rank 1, i.e., the signature of g_0 is $(+,0,0,0)$. This type of family will be called a *degeneration of Lorentz metrics*. It is a family of Lorentz metrics whose light cones open up to hyperplanes. Usually the parameter ε is physically interpretetated as $\varepsilon = c^{-2}$, where c is the speed of light in standard units, so that $c \to \infty$ as $\varepsilon \to 0$.

The family of dual metrics $\{g_\varepsilon^*\}$ has no limit but, in the generic case, the limit contravariant metric $\bar{g}^* := \lim_{\varepsilon\to 0}(-\varepsilon g_\varepsilon^*)$ exists and it has rank 3. We shall only consider degenerations of Lorentz metrics whose limit contravariant metric $\bar{g}^*$ exists and it has rank 3. Therefore, the limit fibre X_0 of a degeneration of Lorentz metrics has a time metric $\bar{g} := g_0 = \lim_{\varepsilon\to 0} g_\varepsilon$ and a space metric $\bar{g}^* := \lim_{\varepsilon\to 0}(-\varepsilon g_\varepsilon^*)$.

The purpose of this paper is to characterize when the limit manifold X_0 inherits the structure of a Newtonian gravitation. For any $\varepsilon > 0$ we consider on the Lorentz manifold $(X_\varepsilon, g_\varepsilon)$ the Levi-Civita connection ∇_ε, the curvature operator $\mathcal{R}_\varepsilon : \Lambda^2 TX_\varepsilon \to \Lambda^2 TX_\varepsilon$ and the contravariant Einstein tensor G_ε^2. We shall show that the limit manifold X_0 is a Newtonian gravitation if and only if there exist the limits of ∇_ε, $\mathcal{R}_\varepsilon$ and G_ε^2 as $\varepsilon \to 0$.

Let us resume the steps of our analysis. The first step is the characterization of the existence of the limit connection. The result is elementary and well known.

Theorem 1.1 (Kunzle [10]). *The limit connection* $\bar{\nabla} := \lim_{\varepsilon\to 0}\nabla_\varepsilon$ *exists if and only if the limit metric* $\bar{g}$ *is locally the square of an exact differential*, (*i.e.*, $\bar{g} = \mathrm{d}t^2$).

In other words, the existence of the limit connection is equivalent to the local existence of an *absolute time* on X_0.

Now let us assume that the limit connection $\bar{\nabla}$ exists. The second step in our analysis determines when the limit manifold $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ is a *Newtonian space–time*, i.e., there exist local coordinates (t, x_1, x_2, x_3) on X_0 such that

$$\bar{g} = \mathrm{d}t \otimes \mathrm{d}t, \qquad \bar{g}^* = \partial_{x_1} \otimes \partial_{x_1} + \partial_{x_2} \otimes \partial_{x_2} + \partial_{x_3} \otimes \partial_{x_3},$$
$$\bar{\nabla}_{\partial_{x_i}} \partial_{x_j} = \bar{\nabla}_{\partial_t} \partial_{x_j} = 0, \quad i, j = 1\text{–}3, \quad \bar{\nabla}_{\partial_t} \partial_t = -(\partial_{x_1} u)\partial_{x_1} - (\partial_{x_2} u)\partial_{x_2} - (\partial_{x_3} u)\partial_{x_3},$$

where $u(t, x_1, x_2, x_3)$ is a smooth function. The coordinate system (t, x_1, x_2, x_3) is said to be a *Newtonian reference frame* and u is the corresponding *potential function*. The differential equations of geodesic lines of $\bar{\nabla}$ are just

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}^2 t} = \frac{\partial u}{\partial x_i},$$

so that they coincide with the Newtonian equations of motion of freely falling bodies. Therefore, the notion of a Newtonian space–time corresponds with the classical Newton theory associated to a potential function. We obtain the following characterization.

Theorem 1.2. *Let us assume that the limit connection $\bar{\nabla}$ exists. Then the following conditions are equivalent*:

(a) $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ *is a Newtonian space–time*.
(b) *The limit curvature operator* $\lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon$ *exists*.
(c) *There exist suitable local coordinates* $(\varepsilon, t, x_1, x_2, x_3)$ *such that*

$$g_\varepsilon = g_{tt}\,\mathrm{d}t^2 + \sum_i g_{ti}(\mathrm{d}t\,\mathrm{d}x_i + \mathrm{d}x_i\,\mathrm{d}t) + \sum_{ij} g_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j,$$

where

$$g_{tt} = 1 - 2u\varepsilon + \mathrm{O}(\varepsilon^2), \qquad g_{ti} = \mathrm{O}(\varepsilon^2), \qquad g_{ij} = -\delta_{ij}\varepsilon + \mathrm{O}(\varepsilon^2),$$

$u(t, x_1, x_2, x_3)$ *being a smooth function*. (*This function u is just the potential function of the limit Newtonian space–time*.)

Note that if we take $\varepsilon = c^{-2}$ in the above expression (and the terms $\mathrm{O}(\varepsilon^2)$ are neglected) then we obtain the metric $g = (1 - 2uc^{-2})\,\mathrm{d}t^2 - c^{-2}\sum \mathrm{d}x_i^2$. This metric is typically used in textbooks to show the Newtonian theory as an approximation of the relativistic theory in a imprecise manner; the above characterization gives a rigorous formulation of this fact.

In the following step of our analysis, we determine when there exists the limit of the contravariant Einstein tensors G^2_ε of g_ε as $\varepsilon \to 0$. This is a central question since the limit $\bar{T}^2 = \lim_{\varepsilon\to 0}(1/8\pi)G^2_\varepsilon$ provides a matter tensor for the limit Newtonian space–time, satisfying the usual dynamical conditions. In such case, one obtains the full structure of a *Newtonian gravitation* on the limit manifold. Our result is the following.

Theorem 1.3. *Let us consider a degeneration of Lorentz metrics such that the limit connection exists and the limit manifold* $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ *is a Newtonian space–time* (*equivalently, the limit curvature operator exists*). *Then the limit of* G^2 *as* $\varepsilon \to 0$ *exists if and only if there exist suitable local coordinates such that*

$$g_{tt} = 1 - 2u\varepsilon + \mathrm{O}(\varepsilon^2), \qquad g_{ti} = \mathrm{O}(\varepsilon^2), \qquad g_{ij} = -\delta_{ij}(\varepsilon + 2u\varepsilon^2) + \mathrm{O}(\varepsilon^2).$$

This expression for the metric g_ε is also obtained by Rendall [16]. He assumes a global condition (flatness at infinity) instead of our local condition on the existence of the limit curvature operator.

The above result has a remarkable application. Using the obtained expression for the metric g_ε, we derived in [13] the Newtonian motion law for n punctual bodies from the field equations, in the following sense. Let us consider a Newtonian gravitation of n punctual *variable* masses following *arbitrary* trajectories. Let us assume that this Newtonian gravitation is the limit of a degeneration of Lorentz metrics. In particular the limit Einstein tensor vanishes, i.e., $\lim_{\varepsilon\to 0} G^2_\varepsilon = 0$, on the complementary set of the punctual masses. Then these punctual masses are constant and their trajectories obey the Newtonian law of motion. This result is contained in the classical EIH papers (see [1] for a clearer exposition) under certain implicit assumptions on the poles of G^2_ε, which are removed in our formulation.

In this paper, any function, manifold, exterior form, etc., is assumed to be of $\mathcal{C}^\infty$-class.

2. Preliminaries: Newtonian gravitation

Definition 2.1. A *Newtonian space–time* is a four-dimensional smooth manifold $\bar{X}$ endowed with a two-covariant symmetric metric $\bar{g}$ (the *time metric*), a two-contravariant symmetric metric $\bar{g}^*$ (the *space metric*) and a torsionless linear connection $\bar{\nabla}$ satisfying the following local conditions.

Axiom 1. At any point of $\bar{X}$ the signature of $\bar{g}^*$ is $(+, 0, 0, 0)$.

Hence, the radical of $\bar{g}$, i.e., the kernel of the polarity $\bar{g} : T_x\bar{X} \to T_x^*\bar{X}$, is a three-dimensional vector space at each point $x \in \bar{X}$. Vectors in this radical are said to be *spatial vectors*.

Locally we have $\bar{g} = \bar{\omega}\otimes\bar{\omega}$ for some one-form $\bar{\omega}$ well-defined up to a sign (for simplicity we do not consider a time orientation). The radical of $\bar{g}$ (i.e., the space of spatial vectors) is the incident space with $\bar{\omega}$.

Axiom 2. At any point of $\bar{X}$ the signature of $\bar{g}^*$ is $(0, +, +, +)$, and the radicals of $\bar{g}^*$ and $\bar{g}$ are mutually incident: $\bar{g}^*(\bar{\omega}) = 0$.

Hence, $\bar{g}^*$ defines an inner product $*$ on spatial vectors:

$$V_1 * V_2 := \bar{g}^*(\omega_1, \omega_2),$$

where ω_i is any one-form such that $\bar{g}^*(\omega_i) = V_i$.

Axiom 3. The parallel transport preserves $\bar{g}$ and $\bar{g}^*$: $\bar{\nabla}\bar{g} = 0$ and $\bar{\nabla}\bar{g}^* = 0$.

The condition $\bar{\nabla}\bar{g} = \bar{\nabla}(\bar{\omega}\otimes\bar{\omega}) = 0$ implies $\mathrm{d}\bar{\omega} = 0$. Therefore, locally we may write $\bar{g} = \mathrm{d}t \otimes \mathrm{d}t$.

Let $\bar{R}(D_1, D_2, D_3) := \bar{\nabla}_{D_1}\bar{\nabla}_{D_2}D_3 - \bar{\nabla}_{D_2}\bar{\nabla}_{D_1}D_3 - \bar{\nabla}_{[D_1,D_2]}D_3$ be the curvature tensor of $\bar{\nabla}$. Let us define the following $(2, 2)$-type tensor

$$\bar{R}_2^2(\omega_1, D_2, \omega_3, D_4) := \omega_3(\bar{R}(D_1, D_2, D_4)), \quad D_1 = \bar{g}^*(\omega_1).$$

Axiom 4. Conservative character of gravitatory forces:

$$\bar{R}_2^2(\omega_1, D_2, \omega_3, D_4) = \bar{R}_2^2(\omega_3, D_4, \omega_1, D_2).$$

Axioms 1–4 define a Newtonian manifold in the sense of Newton–Cartan theory (see [10]). Our concept of a Newtonian space–time requires one more axiom.

Axiom 5. Gyroscope Principle: $\bar{R}(D_1, D_2, V) = 0$ whenever V is a spatial vector.

In particular, this axiom implies the Euclidean character of any spatial slice $t = \text{const}$.

(**2.2**) One may easily show that these five axioms are equivalent to the existence of local coordinates (t, x_1, x_2, x_3) such that

$$\bar{g} = \mathrm{d}t \otimes \mathrm{d}t, \qquad \bar{g}^* = \partial_{x_1} \otimes \partial_{x_1} + \partial_{x_2} \otimes \partial_{x_2} + \partial_{x_3} \otimes \partial_{x_3},$$
$$\bar{\nabla}_{\partial_{x_i}}\partial_{x_j} = \bar{\nabla}_{\partial_t}\partial_{x_j} = 0, \quad i, j = 1\text{–}3, \quad \bar{\nabla}_{\partial_t}\partial_t = -(\partial_{x_1}u)\partial_{x_1} - (\partial_{x_2}u)\partial_{x_2} - (\partial_{x_3}u)\partial_{x_3}$$

for some function $u(t, x_1, x_2, x_2)$, where $\partial_t := \partial/\partial_t$ and $\partial_{x_i} := \partial/\partial x_i$. The above last equality, saying that $\bar{\nabla}_{\partial_t}\partial_t$ is a gradient vector, is a consequence of Axiom 4; this motivates the given denomination for such axiom.

Definition 2.3. A system of local coordinates (t, x_1, x_2, x_3) on a *Newtonian space–time is said to be a Newtonian reference frame* if the equalities (2.2) hold.

Given a Newtonian reference frame, the function $u(t, x_1, x_2, x_3)$ is named the *potential* and its spatial gradient $F := \sum \partial_{x_i}(u)\partial_{x_i} = -\bar{\nabla}_{\partial_t}\partial_t$ is named the *force intensity*. Both concepts depend on the Newtonian reference frame.

(**2.4**) Given a Newtonian space–time, it is easy to check that the Ricci tensor $\bar{R}_2$ of $\bar{\nabla}$ always is proportional to $\bar{g}$. The proportionality coefficient is denoted by $4\pi\rho$:

$$\bar{R}_2 = 4\pi\rho \cdot \bar{g} = 4\pi\rho \cdot \mathrm{d}t^2,$$

and this function ρ is said to be the *mass density*. *Poisson's equation* holds in any Newtonian reference frame:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = -4\pi\rho.$$

Moreover, the differential equations of geodesic lines of $\bar{\nabla}$ are just

$$\frac{\mathrm{d}^2 x_i}{\mathrm{d}^2 t} = \frac{\partial u}{\partial x_i},$$

so that they coincide with the Newtonian equations of motion of freely falling bodies when u is the gravitatory potential.

Definition 2.5. A *matter tensor* on a Newtonian space–time $(\bar{X}, \bar{g}, \bar{g}^*, \bar{\nabla})$ is a two-contravariant symmetric tensor $\bar{T}^2$ such that

(a) $\mathrm{div}_{\bar{\nabla}}\, \bar{T}^2 = 0$,
(b) $\bar{R}_2 = 4\pi \bar{T}_2$,

where $\bar{T}_2(D_1, D_2) := \bar{T}^2(\bar{g}(D_1), \bar{g}(D_2))$.

The condition (b) in Definition 2.5 may be written in the same form as the relativistic Einstein equation:

$$\bar{R}_2 = 8\pi \left(\bar{T}_2 - \tfrac{1}{2}\alpha\bar{g}\right),$$

where α denotes the total contraction of $\bar{g} \otimes \bar{T}^2$ (it coincides with the mass density ρ in the Newtonian case).

(**2.6**) On a Newtonian space–time this condition (b) in Definition 2.5 is equivalent to the equation $\bar{T}^2(\mathrm{d}t, \mathrm{d}t) = \rho$. Now let us consider the vector field $U := (1/\rho)C_1^1(\bar{T}^2 \otimes \mathrm{d}t)$. Then we may write in any Newtonian reference frame

$$\bar{T}^2 = \rho U \otimes U + \sum h_{ij}\partial_{x_i} \otimes \partial_{x_j}, \qquad U = \partial_t + v_1\partial_{x_1} + v_2\partial_{x_2} + v_3\partial_{x_3}$$

for certain smooth functions h_{ij} and v_i. The tensor $\bar{T}^2$ may be interpreted as the matter tensor of a fluid, U being the flow of the fluid, ρ the mass density, and $\sum h_{ij}\partial_{x_i} \otimes \partial_{x_j}$ is the stress tensor. The condition $\mathrm{div}_{\bar{\nabla}}\, \bar{T}^2 = 0$ is equivalent to the pair of classical equations

$$\text{continuity equation}: \quad \mathrm{div}(\rho U) = 0,$$
$$\text{Euler equation}: \quad \rho\bar{\nabla}_U U = -\mathrm{div}\left(\sum h_{ij}\partial_{x_i} \otimes \partial_{x_j}\right).$$

Definition 2.7. A *Newtonian gravitation* is a Newtonian space–time $(\bar{X}, \bar{g}, \bar{g}^*, \bar{\nabla})$ endowed with a matter tensor $\bar{T}^2$.

A *Newtonian perfect fluid* of mass density ρ and pressure p is a Newtonian gravitation whose matter tensor has the form

$$\bar{T}^2 = \rho U \otimes U + p\bar{g}^*.$$

For the sake of completeness, we finish these preliminaries defining the concept of inertial reference frame, although it will not be used in the rest of this paper. Given a Newtonian space–time, a globally defined Newtonian reference frame (t, x_1, x_2, x_3) is said to be *inertial* if its corresponding force intensity F vanishes at infinity or, equivalently, its potential function u coincides with the Newtonian potential defined by the mass density ρ (it is assumed that ρ has compact support at each instant).

It is possible to show the existence of inertial reference frames when the curvature tensor $\bar{R}$ vanishes at infinity.

3. Relative tensors

In this paper we shall use Cartan's exterior differential calculus in a more general setting than it is usual. We shall consider "relative" differential forms with respect to a smooth map $\varphi : X \to Y$ instead of the usual differential forms on a smooth manifold. Now we shall introduce the definitions and we shall state the relative versions of the Poincaré lemma and the Frobenius theorem.

Let us consider a submersion $\varphi : X \to Y$. For any $y \in Y$ we denote $X_y := \varphi^{-1}(y)$. Since φ is a submersion the fibres X_y are subvarieties of X, so that φ may be interpreted as a smooth family of varieties $\{X_y\}$ parametrized by Y.

A tangent vector field D on X is said to be relative if $\varphi_*(D) = 0$, i.e., D is tangent to the fibres X_y of φ. In other words, relative vector fields are just smooth cross-sections of the relative tangent vector bundle $T(X/Y)$ defined by the following exact sequence:

$$0 \to T(X/Y) \to TX \xrightarrow{\varphi_*} \varphi^*(TY) \to 0.$$

More generally, a *relative tensor field* T of type (p, q) on X is a smooth cross-section of the vector bundle

$$T^*(X/Y)^{\otimes^p} \otimes T(X/Y)^{\otimes^q},$$

where $T^*(X/Y)$ denotes the dual bundle of $T(X/Y)$.

A relative tensor field T on X defines by restriction a tensor field $T_{|y}$ on X_y. Note that T is fully determined by the tensors fields $T_{|y}$, so that a relative tensor field T must be viewed as a smooth family $\{T_{|y}\}_{y\in Y}$ of tensor fields on the manifolds $\{X_y\}_{y\in Y}$.

Let us consider a *relative p-form* ω, i.e., a smooth cross-section of $\Lambda^p T^*(X/Y)$. The *exterior differential* of ω is the relative $(p+1)$-form $\mathrm{d}\omega$ defined by the equality: $(\mathrm{d}\omega)_{|y} = \mathrm{d}(\omega_{|y})$.

Given points $x \in X$, $y = \varphi(x) \in Y$, let us denote by $\mathcal{C}_x^\infty$ (resp. $\mathcal{C}_y^\infty$) the ring of germs of smooth functions on X (resp. Y) at x (resp. y). Let Ω_x^p be the $\mathcal{C}_x^\infty$-module of relative p-forms (germs). Let

$$\Omega_x^\bullet = \mathcal{C}_x^\infty \oplus \Omega_x^1 \oplus \Omega_x^2 \oplus \cdots$$

be the differential exterior algebra of the relative forms.

Let J be an ideal of $\mathcal{C}_y^\infty$. Let us consider J as a subset of $\mathcal{C}_x^\infty$ via the natural inclusion $\mathcal{C}_y^\infty \hookrightarrow^{o\varphi} \mathcal{C}_x^\infty$. A relative p-form ω is said to be *exact* modulo (J) if

$$\omega \equiv \mathrm{d}\omega' \bmod (J)$$

for some relative $(p-1)$-form ω', i.e., ω and $\mathrm{d}\omega'$ have the same class in $\Omega_x^p / J \cdot \Omega_x^p \subset \Omega_x^\bullet/(J)$.

A relative p-form ω is said to be *closed* modulo (J) if

$$\mathrm{d}\omega \equiv 0 \bmod (J).$$

Relative Poincaré Lemma 3.1. *A relative p-form $\omega \in \Omega_x^p$ is closed modulo (J) if and only if it is exact modulo (J).*

A *relative Pfaff system* (at x) is a submodule P of Ω_x^1 such that Ω_x^1/P is a free $\mathcal{C}_x^\infty$-module, i.e., P is a direct summand of Ω_x^1. A relative Pfaff system P is said to be integrable if P is generated by exact relative one-forms: $P = \langle \mathrm{d}x_1, \dots, \mathrm{d}x_k \rangle$. More generally, a relative Pfaff system P is said to be *integrable* modulo (J) if P is generated by exact mod (J) relative one-forms.

Relative Frobenius Theorem 3.2. *Let P be a relative Pfaff system. If* $\mathrm{d}P \equiv 0 \bmod (J, P)$, *i.e.,* $\mathrm{d}P \equiv 0$ *in the quotient algebra* $\Omega_x^\bullet/(J, P)$, *then P is integrable modulo* (J),

$$P \equiv \langle \mathrm{d}x_1, \dots, \mathrm{d}x_k \rangle \bmod (J).$$

Moreover, if P is also integrable modulo an ideal $J' \subseteq J$, *then there exist germs* $x_i' \equiv x_i \bmod (J)$ *such that*

$$P \equiv \langle \mathrm{d}x_1', \dots, \mathrm{d}x_k' \rangle \bmod (J').$$

These results have essentially the same proofs that the classical versions have. See details in Appendix A.

4. Degenerations of Lorentz metrics

Let X be a five-dimensional smooth manifold (with boundary) and let $\varepsilon : X \to I = [0, \infty)$ be a submersion. Each fibre X_ϵ on $\epsilon \in I$ is a four-manifold, so that the map $\varepsilon : X \to I$ may be interpreted as a one-parameter smooth family $\{X_\epsilon\}$ of four-manifolds.

Let $\mathcal{D}_X$ be the $C^\infty(X)$-module of all relative vector fields on X. A *relative linear connection* ∇ on X is defined to be a C^∞-linear map

$$\mathcal{D}_X \xrightarrow{\nabla} \mathrm{Hom}_{\mathbb{R}}(\mathcal{D}_X, \mathcal{D}_X), \quad D \mapsto \nabla_D$$

satisfying the usual condition: $\nabla_D(fD') = (Df)D' + f(\nabla_D D')$.

Clearly any relative linear connection ∇ defines, by restriction, a linear connection $\nabla_{|\epsilon}$ on each fibre X_ϵ of $X \to I$, so that a relative connection ∇ must be viewed as a smooth family $\{\nabla_{|\epsilon}\}$ of linear connection on the manifolds $\{X_\epsilon\}$.

Definition 4.1. A relative tensor field T on $\varepsilon : X \to I$, defined on the open subset $\varepsilon > 0$, is said to be *prolongable* to X if it is the restriction of some relative tensor field defined on the total space X. The restriction of this last tensor to the fibre X_0 will be said to be the *limit tensor* of T, and it will denoted by $T_{|0}$.

Definition 4.2. A relative tensor field g on $\varepsilon : X \to I$ is said to be a *degeneration* of Lorentz metrics if

(a) Its restriction $g_{|\epsilon}$ to any fibre X_ϵ, $\epsilon > 0$, is a Lorentz metric, i.e., it has signature $(+, -, -, -)$ at any point of X_ϵ.
(b) Its restriction $g_{|0}$ to X_0 is a metric with signature $(+, 0, 0, 0)$ at any point of X_0.
(c) Let g^* be the relative dual metric, which is defined on the open subset $\varepsilon > 0$. It is assumed that εg^* is prolongable and that its limit $(\varepsilon g)_{|0}$ on X_0 is a metric with signature $(0, -, -, -)$ at any point of X_0.

Therefore, a degeneration is a one-parameter family of Lorentz metrics $\{g_{|\varepsilon}\}$ whose limit metrics $\bar{g} := g_{|0}$, $\bar{g}^* := (-\varepsilon g^*)_{|0}$ are degenerate metrics of respective signature $(+, 0, 0, 0)$ and $(0, +, +, +)$. These metrics $\bar{g}$, $\bar{g}^*$ satisfy Axioms 1 and 2 of a Newtonian space–time. When $\varepsilon > 0$, each fibre X_ε is a relativistic space–time, where $g_{|\varepsilon}$ is the time metric and $-\varepsilon g^*_{|\varepsilon}$ is the space metric, so that the parameter ε may be physically interpreted as $\varepsilon = c^{-2}$, where c is the light speed.

Proposition 4.3. *Let (X, ε, g) be a degeneration of Lorentz metrics. Then the matrix of g in some local basis $\{D_0, D_1, D_2, D_3\}$ of relative vector fields is*

$$\begin{pmatrix} 1 & & & \\ & -\varepsilon & & \\ & & -\varepsilon & \\ & & & -\varepsilon \end{pmatrix}.$$

Proof. Since $(\varepsilon g^*)_{|0}$ has signature $(0, -, -, -)$ at any point of X_0 it is easy to check that there exists a local basis $\{D_0, D_1, D_2, D_3\}$ of relative vector fields on X such that

$$\varepsilon g^* = a\varepsilon D_0 \otimes D_0 - D_1 \otimes D_1 - D_2 \otimes D_2 - D_3 \otimes D_3$$

for some smooth function a on X. Then, in the dual basis $\{\theta_0, \theta_1, \theta_2, \theta_3\}$, we have

$$g = a^{-1}\theta_0 \otimes \theta_0 - \varepsilon\theta_1 \otimes \theta_1 - \varepsilon\theta_2 \otimes \theta_2 - \varepsilon\theta_3 \otimes \theta_3.$$

Since the signature of $g_{|0}$ is $(+,0,0,0)$, we obtain that $a = b\varepsilon^{-1}$, where $a > 0$. Taking $D'_0 = \sqrt{a} D_0$ and $D'_i = D_i$ for $i = 1$–3, we obtain a basis satisfying the required condition. □

Definition 4.4. Let (X, ε, g) be a degeneration of Lorentz metrics. Let $\{D_0, \ldots, D_3\}$ be a basis of relative tangent vectors at a point x and let $\{\theta_0, \ldots, \theta_3\}$ be the corresponding dual basis. These bases are said to be *normal* if

$$g_x = \theta_0 \otimes \theta_0 - \varepsilon(x) \sum_1^3 \theta_i \otimes \theta_i, \qquad (\varepsilon g^*)_x = \varepsilon(x) D_0 \otimes D_0 - \sum_1^3 D_i \otimes D_i.$$

A basis of relative vector field is said to be normal if it is normal at each point. Proposition 4.3 states the local existence of normal bases.

Let (X, ε, g) be a degeneration of Lorentz metrics and let ∇ be the unique relative torsionless linear connection, defined on the open set $\varepsilon > 0$, such that $\nabla g = 0$. The restriction $\nabla_{|\epsilon}$ of ∇ to any fibre X_ϵ, $\epsilon > 0$ coincides with the Levi-Civita connection associated to the corresponding Lorentz metric $g_{|\epsilon}$.

Definition 4.5. We shall say that ∇ is *prolongable* if it is the restriction of a relative linear connection defined on the total space X. The restriction of this connection to the fibre X_0 will be said to be the *limit connection* of ∇, and it will denoted by $\nabla_{|0}$.

We shall give a necessary and sufficient condition for the relative connection ∇ to be prolongable. Previously, let us recall the Cartan structure equations.

(4.6) Let (X, ε, g) be a degeneration of Lorentz metrics. Given a local basis $\{\theta_\alpha\}$ of relative one-forms, let us consider the corresponding connection one-forms $\{\omega_{\alpha\beta}\}$ and curvature two-forms $\{\Omega_{\alpha\beta}\}$ with respect to the relative connection ∇. Of course, these forms are relative and they are defined in the open subset $\varepsilon > 0$. We have the Cartan structure equations (using the standard matrix convention)

$$\mathrm{d}\theta + \omega \wedge \theta = 0, \qquad \mathrm{d}\omega + \omega \wedge \omega = \Omega,$$

and the Bianchi identities

$$\Omega \wedge \theta = 0, \qquad \Omega \wedge \omega - \omega \wedge \Omega = \mathrm{d}\Omega.$$

Moreover, if $\{\theta_\alpha\}$ is a normal basis then we have

$$\omega_{ij} = -\omega_{ji}, \qquad \Omega_{ij} = -\Omega_{ji}, \qquad \omega_{0\alpha} = \varepsilon\omega_{\alpha 0}, \qquad \Omega_{0\alpha} = \varepsilon\Omega_{\alpha 0}$$

(Latin indices shall assume the values 1–3 while Greek indices shall run from 0 to 3).

Proposition 4.7. *Let (X, ε, g) be a degeneration of Lorentz metrics. The relative linear connection ∇ is prolongable if and only if* $\mathrm{d}\theta_0 \equiv 0 \bmod (\varepsilon)$ *for any local normal basis $\{\theta_\alpha\}$ of relative* 1*-forms.*

Proof. If ∇ is prolongable the first structure equation gives

$$\mathrm{d}\theta_0 = -\sum \omega_{0k} \wedge \theta_k = -\varepsilon \sum \omega_{k0} \wedge \theta_k \equiv 0 \bmod (\varepsilon).$$

Reciprocally, if $d\theta_0 \equiv 0 \bmod (\varepsilon)$ we prove that the connection one-forms $\omega_{\alpha\beta}$ are prolongable and, therefore, ∇ is prolongable. Let us write

$$\omega_{\alpha\beta} = \sum A^{\gamma}_{\alpha\beta}\theta_{\gamma}.$$

Let $\{D_\alpha\}$ be the dual basis of $\{\theta_\alpha\}$ and let $\delta_\alpha := g(D_\alpha, D_\alpha)$, i.e., $\delta_0 = 1$ and $\delta_i = -\varepsilon$ for $i = 1$–3. Now the one-forms $\omega_{\alpha\beta}$ are prolongable because of following standard formula for the coefficients $A^{\gamma}_{\alpha\beta}$:

$$\delta_\alpha A^{\gamma}_{\alpha\beta} = \tfrac{1}{2}[\delta_\alpha\, d\theta_\alpha(D_\beta, D_\gamma) - \delta_\gamma\, d\theta_\gamma(D_\alpha, D_\beta) + \delta_\beta\, d\theta_\beta(D_\gamma, D_\alpha)]. \qquad \square$$

From Proposition 4.7 and the Relative Poincaré Lemma 3.1 one obtains directly.

Theorem 4.8 (Kunzle [10]). *The relative linear connection ∇ associated to a degeneration (X, ε, g) is prolongable if and only if there exists locally a smooth function t such that*

$$g \equiv dt^2 \bmod (\varepsilon).$$

(4.9) Now we shall investigate the structure of the limit fibre X_0 of a degeneration (X, ε, g) of Lorentz metrics.

By definition of a degeneration, the limit metrics $\bar{g} := g_{|0}$, $\bar{g}^* := (-\varepsilon g^*)_{|0}$ satisfy Axioms 1 and 2 of a Newtonian space–time.

Now let us also assume that the relative symmetric linear connection ∇ is prolongable, so that it defines a symmetric linear connection $\bar{\nabla} := \nabla_{|0}$ on X_0. On the open subset $\varepsilon > 0$ we have $\nabla g = 0$ and $\nabla(\varepsilon g^*) = \varepsilon(\nabla g^*) = 0$; by continuity, it follows that $\bar{\nabla}\bar{g} = 0$ and $\bar{\nabla}\bar{g}^* = 0$ (Axiom 3).

Finally, let R be the (relative) curvature tensor of ∇ and let us consider the $(2, 2)$-type tensor

$$R^2_2(\omega_1, D_2, \omega_3, D_4) := \omega_3(R(D_1, D_2, D_4)), \qquad D_1 = g^*(\omega_1).$$

This is a relative tensor, defined on the open subset $\varepsilon > 0$, and it satisfies the usual symmetry condition $R^2_2(\omega_1, D_2, \omega_3, D_4) = R^2_2(\omega_3, D_4, \omega_1, D_2)$. It is clear that $-\varepsilon R^2_2$ is prolongable, because so are $-\varepsilon g^*$ and R. Hence, by continuity, the tensor $\bar{R}^2_2 = (-\varepsilon R^2_2)_{|0}$ on X_0 also satisfies the above symmetry condition (Axiom 4).

In conclusion, we have proved the following.

Proposition 4.10. *Let (X, ε, g) be a degeneration of Lorentz metrics. If the relative connection ∇ is prolongable, then the limit fibre $(X_0, \bar{g} := g_{|0}, \bar{g}^* := (-\varepsilon g^*)_{|0}, \bar{\nabla} := \nabla_{|0})$ satisfies all the axioms of a Newtonian space–time, except for the fifth one (Gyroscope Principle).*

5. Limit of the curvature operator

It is necessary to impose a supplementary condition on a degeneration (X, ε, g) to achieve the limit fibre $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ satisfies the Gyroscope Principle. In this section we shall examine this point.

(**5.1**) Let us consider the relative *curvature operator*, which is obtained by rising the last covariant index in the curvature tensor,

$$\mathcal{R}(D_1, D_2, \omega_3, \omega_4) := \omega_3(\nabla_{D_1}\nabla_{D_2}D_4 - \nabla_{D_2}\nabla_{D_1}D_4 - \nabla_{[D_1,D_2]}D_4),$$

where $D_4 := g^*(\omega_4)$. The curvature operator $\mathcal{R}$ is defined on the open subset $\varepsilon > 0$. The following result characterizes the prolongability of $\mathcal{R}$ in terms of the curvature two-forms Ω_{ij}.

Lemma 5.2. *Let (X, ε, g) be a degeneration of Lorentz metrics whose relative connection ∇ is prolongable. Then*

(a) *The limit $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ is a Newtonian space–time if and only if $\Omega_{ij} \equiv 0 \bmod (\varepsilon)$ for any normal basis $\{D_\alpha\}$.*
(b) *The relative curvature operator $\mathcal{R}$ is prolongable if and only if $\Omega_{ij} \equiv 0 \bmod (\varepsilon)$ for any normal basis $\{D_\alpha\}$.*

(recall that we use Latin indices for 1–3 and Greek indices for 0–3).

Proof.

(a) Let us consider the relative curvature tensor R as a (3, 1)-type tensor:

$$R(E_1, E_2, \omega_3, E_4) = \omega_3(\nabla_{E_1}\nabla_{E_2}E_4 - \nabla_{E_2}\nabla_{E_1}E_4 - \nabla_{[E_1,E_2]}E_4).$$

Let $\{\theta_\alpha\}$ be the dual basis of $\{D_\alpha\}$. The expression of R with respect to these bases is

$$R = \sum_{\alpha\beta} \Omega_{\alpha\beta} \otimes D_\alpha \otimes \theta_\beta.$$

Then the condition $\Omega_{ij} \equiv 0 \bmod (\varepsilon)$ for all indexes i, j is equivalent to the Gyroscope Principle: $R(-, -, -, D_j) \equiv 0 \bmod (\varepsilon)$ for all index j (recall that $\Omega_{0j} = \varepsilon\Omega_{j0} \equiv 0$). Using Proposition 4.10 the proof finishes.

(b) The result follows from the expression of the curvature operator in any normal basis:

$$\mathcal{R} = \sum_{\alpha\beta} \frac{1}{\delta_\beta} \Omega_{\alpha\beta} \otimes D_\alpha \otimes D_\beta,$$

where $\delta_0 = 1$ and $\delta_i = -\varepsilon$. □

As a direct consequence we have the following.

Proposition 5.3. *Let (X, ε, g) be a degeneration of Lorentz metrics whose relative connection ∇ is prolongable. Then the limit $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ is a Newtonian space–time if and only if the curvature operator $\mathcal{R}$ is prolongable.*

In conclusion, the limit fibre of a degeneration of Lorentz metrics inherits the structure of a Newtonian space–time if and only if the relative connection and the curvature operator are prolongable.

Let (X, ε, g) be a degeneration of Lorentz metrics. If the relative connection is prolongable we have showed in Theorem 4.8 that $g \equiv \mathrm{d}t^2 \bmod (\varepsilon)$. Now, we shall obtain the Taylor expansion of g up to order 2 in ε, when the connection and the curvature operator are assumed to be prolongable.

Theorem 5.4. *Let (X, ε, g) be a degeneration of Lorentz metrics whose connection ∇ and curvature operator $\mathcal{R}$ are prolongable. Then there exists local coordinates $(\varepsilon, t, x_1, x_2, x_3)$ such that*

$$g \equiv (1 - 2u\varepsilon)\,\mathrm{d}t^2 - \varepsilon \sum \mathrm{d}x_i^2 \bmod (\varepsilon^2),$$

where u is a smooth function on the coordinates (t, x_1, x_2, x_3).

Proof. Let $\mathcal{N}(X) \to X$ be the bundle of normal bases (the fibre of this bundle over a point $x \in X$ is the six-manifold of all normal bases at x). Let $\{\theta_0, \dots, \theta_3\}$ be the universal normal basis of one-forms on $\mathcal{N}(X)$. By Proposition 4.7 and Lemma 5.2b, we know that

$$\mathrm{d}\theta_0 \equiv 0, \qquad \Omega_{ij} \equiv 0 \bmod (\varepsilon)$$

(recall that we use Latin indices for 1–3).

Using the second structure equation and the Relative Frobenius Theorem 3.2 one checks immediately that the Pfaff system $\langle \omega_{12}, \omega_{13}, \omega_{23} \rangle$ is integrable $\bmod (\varepsilon)$.

Moreover, the first Bianchi identity gives

$$\Omega_{i0} \wedge \theta_0 = 0 \bmod (\varepsilon),$$

i.e., $\Omega_{i0} \equiv \alpha_i \wedge \theta_0 \bmod (\varepsilon)$ for some one-form α_i. The second Bianchi identity gives

$$\mathrm{d}\Omega_{i0} \equiv 0 \bmod (\varepsilon, \omega_{12}, \omega_{13}, \omega_{23}),$$

hence $\mathrm{d}\alpha_i \equiv 0 \bmod (\varepsilon, \theta_0, \omega_{12}, \omega_{13}, \omega_{23})$. The Relative Poincaré Lemma A.5′ implies $\alpha_i \equiv \mathrm{d}f_i \bmod (\varepsilon, \theta_0, \omega_{12}, \omega_{13}, \omega_{23})$ for some function f_i, and then

$$(*) \quad \Omega_{i0} \equiv \mathrm{d}f_i \wedge \theta_0 \bmod (\varepsilon, \omega_{12}, \omega_{13}, \omega_{23}).$$

Now we may prove that the following Pfaff system

$$P = \langle \omega_{10} - f_1\theta_0, \omega_{20} - f_2\theta_0, \omega_{30} - f_i\theta_0, \omega_{12}, \omega_{13}, \omega_{23} \rangle$$

is integrable $\bmod (\varepsilon)$. From the second structure equation we have

$$\mathrm{d}(\omega_{i0} - f_i\theta_0) \equiv \Omega_{i0} - \sum_j \omega_{ij} \wedge \omega_{j0} - \mathrm{d}f_i \wedge \theta_0 \overset{*}{\equiv} 0 \bmod (\varepsilon, P),$$

and by the Relative Frobenius Theorem 3.2 we conclude that P is integrable $\bmod (\varepsilon)$.

Therefore, restricting our attention to a suitable subvariety of $\mathcal{N}(X)$ of codimension 6 (which corresponds with a section of the bundle $\mathcal{N}(X) \to X$) we get a (local) normal vector field basis on X such that

$$\omega_{i0} \equiv f_i\theta_0, \qquad \omega_{ij} \equiv 0 \bmod (\varepsilon)$$

for certain smooth functions f_i on X.

Now, the first structure equation gives us that $\mathrm{d}\theta_i \equiv 0 \bmod (\varepsilon)$, hence we may write locally

$$\theta_i \equiv \mathrm{d}x_i \bmod (\varepsilon).$$

On the other hand, we have

$$\mathrm{d}\theta_0 = -\sum \omega_{0i} \wedge \theta_i = -\varepsilon \sum \omega_{i0} \wedge \theta_i \equiv -\varepsilon \sum f_i \theta_0 \wedge \theta_i \equiv \varepsilon \sum f_i \theta_i \wedge \theta_0 \bmod (\varepsilon^2).$$

Differentiating we obtain $0 = \mathrm{d}\mathrm{d}\theta_0 \equiv \varepsilon\, \mathrm{d}(\sum f_i\theta_i) \wedge \theta_0 \bmod (\varepsilon^2)$, hence $\mathrm{d}(\sum f_i\theta_i) \equiv 0 \bmod (\varepsilon, \theta_0)$ and then $\sum f_i\theta_i \equiv \mathrm{d}f \bmod (\varepsilon, \theta_0)$ for some smooth function f on X. In conclusion, we have obtained that

$$\mathrm{d}\theta_0 \equiv \varepsilon\, \mathrm{d}f \wedge \theta_0 \bmod (\varepsilon^2).$$

This implies that

$$\mathrm{d}(\mathrm{e}^{-\varepsilon f}\theta_0) \equiv 0 \bmod (\varepsilon^2),$$

so we may write

$$\mathrm{e}^{-\varepsilon f}\theta_0 \equiv \mathrm{d}t \bmod (\varepsilon^2)$$

for certain smooth function t, i.e.,

$$\theta_0 \equiv \mathrm{e}^{\varepsilon f}\,\mathrm{d}t \equiv (1+\varepsilon f)\,\mathrm{d}t \bmod (\varepsilon^2).$$

Taking $\tilde{u} = -f$ we have:

$$g = \theta_0^2 - \varepsilon \sum \theta_i^2 \equiv (1 - 2\tilde{u}\varepsilon)\,\mathrm{d}t^2 - \varepsilon \sum \mathrm{d}x_i^2 \bmod (\varepsilon^2).$$

Since $\tilde{u} = u + \varepsilon v$ for some smooth function $u(t, x_1, x_2, x_3)$, we finish the proof. □

Remark 5.5. A direct computation shows that the inverse of Theorem 5.4 holds. *For any degeneration of the form* $g \equiv (1 - 2u\varepsilon^r)\,\mathrm{d}t^2 - \varepsilon \sum \mathrm{d}x_i^2 \bmod (\varepsilon^2)$ *the relative connection* ∇ *and the curvature operator* $\mathcal{R}$ *are prolongable*. Then, by Proposition 5.3, the limit $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ is a Newtonian space–time. Moreover, the coordinates (t, x_1, x_2, x_3) are a Newtonian reference frame (Definition 2.3) on X_0 and u is the corresponding potential function.

As a consequence, it results that *any Newtonian space–time is locally the limit of some degeneration of Lorentz metrics*.

6. Newtonian limit of the Einstein tensor

In the relativistic case, the geometric structure of space–time determines the energy-impulse tensor via the Einstein equation

$$T^2 = \frac{1}{8\pi} G^2,$$

where G^2 is the two-contravariant Einstein tensor. Given a degeneration (X, ε, g) of Lorentz metrics, it is natural to ask when the relative Einstein tensor G^2 is prolongable. We shall examine this question in this section. First, we shall show that if G^2 is prolongable then

$$\bar{T}^2 := \frac{1}{8\pi} G^2_{|0}$$

defines a matter tensor (Definition 2.5) on the limit fibre X_0.

Let us consider a degeneration of Lorentz metrics (X, ε, g) whose relative contravariant Einstein tensor G_2 is prolongable. Then the Einstein operator $G := C_1^1(G^2 \otimes g)$ and the covariant Einstein tensor G^2 are prolongable. Therefore, the scalar curvature $r = -\text{trace}(G)$, the Ricci operator $\text{Ric} = G + (r/2)\text{Id}$, and the Ricci tensor $R_2 = G_2 + (r/2)g$ are also prolongable.

Proposition 6.1. *Let (X, ε, g) be a degeneration of Lorentz metrics such that the contravariant Einstein tensor G^2 is prolongable. Then we have the following facts in the limit fibre X_0:*

(a) *The limit Ricci tensor $\bar{R}_2$ is proportional to $\bar{g}$. As in* (2.4) *we write $\bar{R}_2 = 4\pi\rho \cdot \bar{g}$.*
(b) *The limit scalar curvature is $\bar{r} = -8\pi\rho$.*
(c) *The limit covariant Einstein tensor is $\bar{G}_2 = 8\pi\rho \cdot \bar{g}$.*
(d) *If ∇ is prolongable then $\text{div}_{\bar{\nabla}}\, \bar{G}^2 = 0$. (Therefore, $\bar{T}^2 := (1/8\pi)\bar{G}^2$ is a matter tensor on X_0.)*

Proof.

(a) Let us write $\bar{g} = \bar{\omega} \otimes \bar{\omega}$ for some one-form $\bar{\omega}$ on X_0. Since

$$(\text{matrix}\, \bar{R}_2) = (\text{matrix}\, \overline{\text{Ric}}) \circ (\text{matrix}\, \bar{g}),$$

we conclude that $\bar{R}_2$ is proportional to $\bar{\omega} \otimes \bar{\omega}$.

(b) Since

$$(\text{matrix}\, \bar{G}) = (\text{matrix}\, \bar{G}^2) \circ (\text{matrix}\, \bar{g}),$$

we obtain that $\text{rad}\, \bar{g} \subseteq \ker \bar{G}$, i.e., $\bar{G} = D \otimes \bar{\omega}$ for certain vector field D on X_0. Then $\bar{r} = -\text{trace}(\bar{G}) = -\bar{\omega}(D)$. Moreover,

$$\begin{aligned} 4\pi\rho\bar{\omega} \otimes \bar{\omega} = \bar{R}_2 &= C_1^1(\bar{g} \otimes \overline{\text{Ric}}) = C_1^1(\bar{\omega} \otimes \bar{\omega} \otimes (D \otimes \bar{\omega} + \tfrac{1}{2}(\bar{r})\text{Id})) \\ &= (\bar{\omega}(D) + \tfrac{1}{2}(\bar{r}))\bar{\omega} \otimes \bar{\omega} = -\tfrac{1}{2}(\bar{r})\bar{\omega} \otimes \bar{\omega}, \end{aligned}$$

hence $\bar{r} = -8\pi\rho$.

(c) $G_2 = C_1^1(\bar{g} \otimes \bar{G}) = C_1^1(\bar{\omega} \otimes \bar{\omega} \otimes D \otimes \bar{\omega}) = \bar{\omega}(D)\bar{\omega} \otimes \bar{\omega} = -\bar{r}\bar{g} = 8\pi\rho\bar{g}$.

(d) Finally, the contracted Bianchi identity $\text{div}_\nabla G^2 = 0$ induces, by continuity, the identity $\text{div}_{\bar{\nabla}}\, \bar{G}^2 = 0$ in the limit fibre. □

As a direct consequence of Propositions 5.3 and 6.1 we obtain the following.

Theorem 6.2. *Let (X, ε, g) be a degeneration of Lorentz metrics. Let us assume that the connection ∇, the curvature operator $\mathcal{R}$ and the Einstein tensor G^2 are prolongable. Then the limit fibre $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla}, \bar{T}^2)$ is a Newtonian gravitation.*

If one drops the assumption on the curvature operator in the above statement then one obtains the following result (consequence of Propositions 4.10 and 6.1).

Theorem 6.2′. *Let (X, ε, g) be a degeneration of Lorentz metrics. Let us assume that the connection ∇ and the Einstein tensor G^2 are prolongable. Then the limit fibre $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla}, \bar{T}^2)$ satisfies all the axioms of a Newtonian gravitation, except for the fifth, and its Ricci tensor $\bar{R}_2$ is proportional to $\bar{g}$.*

Remark 6.3. Under the hypotheses of the above result, the limit fibre is very near to be a Newtonian gravitation. In fact, by a result of Kunzle [9, Theorem 10] the fifth axiom of a Newtonian space–time may be deduced from the condition "$\bar{R}_2$ proportional to $\bar{g}$" if it is assumed that the connection $\bar{\nabla}$ is flat at infinity. In conclusion, one obtains the following result. *Let (X, ε, g) be a degeneration of Lorentz metrics such that the connection ∇ and the Einstein tensor G^2 are prolongable, and ∇ is "flat at infinity". Then the limit fibre $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla}, \bar{T}^2)$ is a Newtonian gravitation.*

Now we shall characterize the prolongability of the Einstein tensor G^2 in terms of the Taylor expansion of g in ε. We begin with the following.

Lemma 6.4. *Let (X, ε, g) be a degeneration of Lorentz metrics and let $\{\theta_0, \dots, \theta_3\}$ be a normal basis of relative 1-forms. The Einstein tensor G^2 is prolongable if and only if the following conditions hold*

$$\Omega_{12} \wedge \theta_3 - \Omega_{13} \wedge \theta_2 + \Omega_{23} \wedge \theta_1 \equiv 0 \bmod (\varepsilon),$$
$$\Omega_{0i} \wedge \theta_j - \Omega_{0j} \wedge \theta_i + \Omega_{ij} \wedge \theta_0 \equiv 0 \bmod (\varepsilon^2)$$

for any $1 \le i < j \le 3$.

Proof. The computation is simplified using a certain important form Θ introduced by Cartan [3]. Let us consider the identity operator $\mathrm{Id} : T(X/I) \to T(X/I)$ as a relative one-form valued in the module of relative vector fields, and the curvature operator $\mathcal{R}$ as a relative two-form valued in the module of sections of $\Lambda^2 T(X/I)$. Then $\mathrm{Id} \wedge \mathcal{R}$ is a three-form valued in $\Lambda^3 T(X/I)$. Applying the Hodge's star we finally obtain a relative three-form valued in the module of relative vector fields

$$\Theta := (\mathrm{Id} \wedge \mathcal{R})^*.$$

This form and the relative Einstein tensor are mutually determined by the formula

$$\Theta = C_1^1(\omega_X \otimes G^2),$$

where $\omega_X = \varepsilon^{3/2}\theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3$ is the relative volume form. The above formula implies that G^2 is prolongable if and only if $\varepsilon^{-3/2}\Theta$ is prolongable.

Now we examine the prolongability of $\varepsilon^{-3/2}\Theta$. We have

$$\mathcal{R} = -\sum_{\alpha<\beta} \varepsilon^{-1}\Omega_{\alpha\beta} \otimes D_\alpha \wedge D_\beta,$$

$$*(D_1 \wedge D_2 \wedge D_3) = -\varepsilon^{3/2}D_0, \qquad *(D_0 \wedge D_2 \wedge D_3) = -\varepsilon^{1/2}D_1,$$
$$*(D_0 \wedge D_1 \wedge D_3) = +\varepsilon^{1/2}D_2, \qquad *(D_0 \wedge D_1 \wedge D_2) = -\varepsilon^{1/2}D_3.$$

Therefore,

$$\begin{aligned}\varepsilon^{-3/2}\Theta = \varepsilon^{-3/2}(\mathrm{Id} \wedge \mathcal{R})^* &= -\sum_\gamma \sum_{\alpha<\beta} \varepsilon^{-5/2}(\theta_\gamma \wedge \Omega_{\alpha\beta}) \otimes *(D_\gamma \wedge D_\alpha \wedge D_\beta) \\ &= \varepsilon^{-1}(\theta_1 \wedge \Omega_{23} - \theta_2 \wedge \Omega_{13} + \theta_3 \wedge \Omega_{12}) \otimes D_0 \\ &\quad + \varepsilon^{-2}(\theta_0 \wedge \Omega_{23} - \theta_2 \wedge \Omega_{03} + \theta_3 \wedge \Omega_{02}) \otimes D_1 \\ &\quad - \varepsilon^{-2}(\theta_0 \wedge \Omega_{13} - \theta_1 \wedge \Omega_{03} + \theta_3 \wedge \Omega_{01}) \otimes D_2 \\ &\quad + \varepsilon^{-2}(\theta_0 \wedge \Omega_{12} - \theta_1 \wedge \Omega_{02} + \theta_2 \wedge \Omega_{01}) \otimes D_3,\end{aligned}$$

and the result follows. □

Theorem 6.5. *Let (X, ε, g) be a degeneration of Lorentz metrics such that the relative connection ∇, the curvature operator $\mathcal{R}$ and the Einstein tensor G^2 are prolongable. Then there exists local coordinates $(\varepsilon, t, x_1, x_2, x_3)$ on X such that the components of g are*

$$g_{tt} \equiv 1 - 2u\varepsilon \bmod (\varepsilon^2), \qquad g_{ti} \equiv 0 \bmod (\varepsilon^2), \qquad g_{ij} \equiv -(\varepsilon + 2u\varepsilon^2)\delta_{ij} \bmod (\varepsilon^3),$$

where u is a smooth function on the coordinates (t, x_1, x_2, x_3).

Proof. As in the proof of Theorem 5.4 we consider the bundle of normal bases $\mathcal{N}(X) \to X$ and the universal normal basis $\{\theta_0, \dots, \theta_3\}$ of one-forms on $\mathcal{N}(X)$. In the proof of Theorem 5.4 we have showed the following facts:

(a) The Pfaff system $\langle\omega_{12}, \omega_{13}, \omega_{23}\rangle$ is integrable mod (ε).
(b) $\Omega_{i0} \equiv \mathrm{d}f_i \wedge \theta_0 \bmod (\varepsilon, \omega_{12}, \omega_{13}, \omega_{23})$.
(c) The Pfaff system $\langle\omega_{10} - f_1\theta_0, \omega_{20} - f_2\theta_0, \omega_{30} - f_3\theta_0\rangle$ is integrable mod $(\varepsilon, \omega_{12}, \omega_{13}, \omega_{23})$.

Moreover, the prolongability of the Einstein tensor G^2 implies by Lemma 6.4 that

$$\begin{aligned}\Omega_{ij} \wedge \theta_0 \equiv \Omega_{0j} \wedge \theta_i - \Omega_{0i} \wedge \theta_j &\equiv \varepsilon(\Omega_{j0} \wedge \theta_i - \Omega_{i0} \wedge \theta_j) \\ &\overset{b}{\equiv} \varepsilon(\mathrm{d}f_i \wedge \theta_j - \mathrm{d}f_j \wedge \theta_i) \wedge \theta_0\end{aligned}$$

mod $(\varepsilon^2, \varepsilon\omega_{12}, \varepsilon\omega_{13}, \varepsilon\omega_{23})$ for any $1 \le i \le j \le 3$. Therefore,

$$\Omega_{ij} \equiv \varepsilon(\mathrm{d}f_i \wedge \theta_j - \mathrm{d}f_j \wedge \theta_i)$$

mod $(\varepsilon^2, \varepsilon\omega_{12}, \varepsilon\omega_{13}, \varepsilon\omega_{23}, \theta_0)$. Since $\Omega_{ij} \equiv 0 \bmod (\varepsilon)$ (by Lemma 5.2b) we conclude that

(d) $\Omega_{ij} \equiv \varepsilon(\mathrm{d}f_i \wedge \theta_j - \mathrm{d}f_j \wedge \theta_i) \bmod (\varepsilon^2, \varepsilon\omega_{12}, \varepsilon\omega_{13}, \varepsilon\omega_{23}, \varepsilon\theta_0)$.

From (c), restricting our attention to a suitable subvariety of $\mathcal{N}(X)$ (of codimension 3), we may assume that

(e) $\omega_{i0} \equiv f_i\theta_0 \bmod (\varepsilon, \omega_{12}, \omega_{13}, \omega_{23})$.

Now we may prove that the following Pfaff system of rank 3

$$P = \langle \omega_{ij} - \varepsilon(f_i\theta_j - f_j\theta_i)\rangle, \quad 1 \le i \le j \le 3$$

is integrable mod $(\varepsilon^2, \varepsilon\theta_0)$. In fact, modulo $(\varepsilon^2, \varepsilon\theta_0, P)$ we have

$$\begin{aligned}\mathrm{d}\omega_{ij} &= \Omega_{ij} - \omega_{i0} \wedge \omega_{0j} - \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} - \omega_{i0} \wedge \varepsilon\omega_{j0} - \sum_k \omega_{ik} \wedge \omega_{kj} \\ &\overset{e}{\equiv} \Omega_{ij} - f_i\theta_0 \wedge \varepsilon f_j\theta_0 - \sum_k \varepsilon(f_i\theta_k - f_k\theta_i) \wedge \varepsilon(f_k\theta_j - f_j\theta_k) \\ &\equiv \Omega_{ij} \overset{d}{\equiv} \varepsilon(\mathrm{d}f_i \wedge \theta_j - \mathrm{d}f_j \wedge \theta_i),\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}(\varepsilon f_i\theta_j) &= \varepsilon\,\mathrm{d}f_i \wedge \theta_j - \varepsilon f_i\left(\omega_{j0} \wedge \theta_0 + \sum_k \omega_{jk} \wedge \theta_k\right) \\ &\equiv \varepsilon\,\mathrm{d}f_i \wedge \theta_j - \varepsilon f_i\left(\omega_{j0} \wedge \theta_0 + \sum_k \varepsilon(f_j\theta_k - f_k\theta_j) \wedge \theta_k\right) \equiv \varepsilon\,\mathrm{d}f_i \wedge \theta_j,\end{aligned}$$

hence $\mathrm{d}(\omega_{ij} - \varepsilon(f_i\theta_j - f_j\theta_i)) \equiv 0 \bmod (\varepsilon^2, \varepsilon\theta_0, P)$. By Proposition A.4 we conclude that P is integrable mod $(\varepsilon^2, \varepsilon\theta_0)$. Therefore, restricting our attention to a suitable section of $\mathcal{N}(X) \to X$, we may obtain a normal basis satisfying

$$\omega_{i0} \equiv f_i\theta_0, \qquad \omega_{ij} \equiv \varepsilon(f_i\theta_j - f_j\theta_i) \bmod (\varepsilon^2, \varepsilon\theta_0)$$

for any $i, j > 0$.

Now, the argument given in the last part of the proof of Theorem 5.4 shows that

$$\sum_k f_k\theta_k \equiv \mathrm{d}f \bmod (\varepsilon, \theta_0), \quad \theta_0 \equiv \mathrm{e}^{\varepsilon f}\,\mathrm{d}t \bmod (\varepsilon^2)$$

for certain functions f, t. Taking $\tilde{u} := -f$, it results

$$(*) \quad \theta_0 \equiv (1 - \varepsilon\tilde{u})\,\mathrm{d}t \bmod (\varepsilon^2).$$

Moreover,

$$\begin{aligned}\mathrm{d}\theta_i &= -\omega_{i0} \wedge \theta_0 - \sum_k \omega_{ik} \wedge \theta_k \equiv -f_i\theta_0 \wedge \theta_0 - \sum_k \varepsilon(f_i\theta_k - f_k\theta_i) \wedge \theta_k \\ &\equiv -\sum_k \varepsilon f_k\theta_k \wedge \theta_i \equiv -\varepsilon\,\mathrm{d}f \wedge \theta_i \equiv \varepsilon\,\mathrm{d}\tilde{u} \wedge \theta_i \bmod (\varepsilon^2, \varepsilon\theta_0),\end{aligned}$$

hence $\mathrm{d}(\mathrm{e}^{-\varepsilon\tilde{u}}\theta_i) \equiv 0 \bmod (\varepsilon^2, \varepsilon\theta_0)$. By Proposition A.6 we have $\mathrm{e}^{-\varepsilon\tilde{u}}\theta_i \equiv \mathrm{d}x_i \bmod (\varepsilon^2, \varepsilon\theta_0)$ for certain functions x_i, and then

$$(**) \quad \theta_i \equiv \mathrm{e}^{\varepsilon\tilde{u}}\,\mathrm{d}x_i \equiv (1+\tilde{u}\varepsilon)\,\mathrm{d}x_i \bmod (\varepsilon^2, \varepsilon\theta_0) = (\varepsilon^2, \varepsilon\,\mathrm{d}t).$$

The proof concludes substituting $(*)$ and $(**)$ in the equality $g = \theta_0^2 - \varepsilon\sum_k \theta_k^2$. □

Remark 6.6. A similar result to Theorem 6.5 is obtained by Rendall [16]. He assumes a global condition (flatness at infinity) instead of our local condition on the prolongability of the curvature operator.

Remark 6.7. A direct computation shows that the inverse of Theorem 6.5 holds. *Let us consider a degeneration* $g = g_{tt}\,\mathrm{d}t^2 + \sum_i g_{ti}(\mathrm{d}t\,\mathrm{d}x_i + \mathrm{d}x_i\,\mathrm{d}t) + \sum_{ij} g_{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j$, *where*

$$g_{tt} \equiv 1 - 2u\varepsilon \bmod (\varepsilon^2), \qquad g_{ti} \equiv 0 \bmod (\varepsilon^2), \qquad g_{ij} \equiv -(\varepsilon + 2u\varepsilon^2)\delta_{ij} \bmod (\varepsilon^3),$$

and u *is a smooth function on the coordinates* (t, x_1, x_2, x_3). *Then the relative connection* ∇, *the curvature operator* $\mathcal{R}$ *and the Einstein tensor* G^2 *are prolongable. By Theorem 6.2, the limit* $(X_0, \bar{g}, \bar{g}^*, \bar{T}^2)$ *is a Newtonian gravitation. Moreover, the coordinates* (t, x_1, x_2, x_3) *are a Newtonian reference frame (Definition 2.3) on* X_0 *and* u *is the corresponding potential function.*

Let us sketch the computation. From the hypothesis on g, one obtains a normal basis $\{\theta_0, \theta_1, \theta_2, \theta_3\}$ such that

$$\theta_0 \equiv \mathrm{e}^{\varepsilon f}\,\mathrm{d}t \bmod (\varepsilon^2), \qquad \theta_i \equiv \mathrm{e}^{\varepsilon f}\,\mathrm{d}x_i \bmod (\varepsilon^2, \varepsilon\,\mathrm{d}t),$$

where $f = -u$. By Proposition 4.7 we know that the relative connection is prolongable.

Let $\mathrm{d}f \equiv \sum_i f_i\theta_i \bmod (\theta_0)$. Computing the connection one-forms and the curvature two-forms, one obtains

$$\omega_{i0} \equiv f_i\theta_0 \bmod (\varepsilon), \qquad \omega_{ij} \equiv \varepsilon(f_i\theta_j - f_j\theta_i) \bmod (\varepsilon^2, \varepsilon\theta_0),$$

$$\Omega_{i0} \equiv \mathrm{d}f_i \wedge \theta_0 \bmod (\varepsilon), \qquad \Omega_{ij} \equiv \varepsilon(\mathrm{d}f_i \wedge \theta_j - \mathrm{d}f_j \wedge \theta_i) \bmod (\varepsilon^2, \varepsilon\theta_0).$$

Using Proposition 5.2b and Lemma 6.4 we conclude that the curvature operator $\mathcal{R}$ and the Einstein tensor G^2 are prolongable.

Remark 6.8. The expression of g given in Theorem 6.5 is equivalent to the following pair of equations:

$$g \equiv (1-2u\varepsilon)\,\mathrm{d}t \otimes \mathrm{d}t - \varepsilon\sum \mathrm{d}x_i \otimes \mathrm{d}x_i \bmod (\varepsilon^2),$$

$$-\varepsilon g^* \equiv -\varepsilon\partial_t \otimes \partial_t + (1-2u\varepsilon)\sum \partial_{x_i} \otimes \partial_{x_i} \bmod (\varepsilon^2).$$

Remark 6.9. By means of a suitable change of coordinates, we may assume that the coordinates (t, x_1, x_2, x_3) of Theorem 6.5 coincide on X_0 with an arbitrary Newtonian reference frame. In other words, under the hypothesis of Theorem 6.5, any Newtonian reference frame (t, x_1, x_2, x_3) at a point $x_0 \in X_0$ may be locally extended so as obtain a local coordinate system $(\varepsilon, t, x_1, x_2, x_3)$ on X such that the components of g are

$$g_{tt} \equiv 1 - 2u\varepsilon \bmod (\varepsilon^2), \qquad g_{ti} \equiv 0 \bmod (\varepsilon^2), \qquad g_{ij} \equiv -(\varepsilon + 2u\varepsilon^2)\delta_{ij} \bmod (\varepsilon^3),$$

where $u(t, x_1, x_2, x_3)$ is the potential function of the given Newtonian reference frame.

See Appendix B for a global version of Remark 6.9.

7. Examples

7.1. Degeneration of Schwarzschild metrics

Let us consider the Lorentz metric corresponding to an isolated mass m in standard units:

$$(1 - 2r^{-1}mc^{-2})\,\mathrm{d}t^2 - c^{-2}(1 - 2r^{-1}mc^{-2})^{-1}\,\mathrm{d}r^2 - c^{-2}r^2(\mathrm{d}\alpha^2 + (\sin^2\alpha)\,\mathrm{d}\beta^2),$$

where c is the speed of light and r must be understood as the "distance" to such mass. Taking $\varepsilon = c^{-2}$ we obtain the following degeneration of Lorentz metrics:

$$g = (1 - 2r^{-1}m\varepsilon)\,\mathrm{d}t^2 - \varepsilon(1 - 2r^{-1}m\varepsilon)^{-1}\,\mathrm{d}r^2 - \varepsilon r^2(\mathrm{d}\alpha^2 + (\sin^2\alpha)\,\mathrm{d}\beta^2).$$

Taking $u = m/r$ we have

$$\begin{aligned} g &\equiv (1 - 2u\varepsilon)\,\mathrm{d}t^2 - \varepsilon(\mathrm{d}r^2 + r^2\,\mathrm{d}\alpha^2 + r^2(\sin^2\alpha)\,\mathrm{d}\beta^2) \\ &= (1 - 2u\varepsilon)\,\mathrm{d}t^2 - \varepsilon\sum \mathrm{d}x_i^2 \bmod (\varepsilon^2). \end{aligned}$$

By Remark 5.5, the connection and the curvature operator are prolongable and the limit fibre is a Newtonian space–time. Moreover, (t, x_1, x_2, x_3) is a Newtonian reference frame on this Newtonian space–time, $u = m/r$ being the corresponding potential (it is the Newtonian gravitation corresponding to a punctual mass). Note that the Einstein tensor is obviously prolongable since it is null.

7.2. Degeneration of Friedmann–Robertson–Walker metrics

Let us consider the Robertson–Walker metric

$$\mathrm{d}t^2 - r(t)^2\,\mathrm{d}\sigma^2,$$

where $\mathrm{d}\sigma^2$ is the three-dimensional Riemannian metric of constant curvature K, i.e.,

$$\mathrm{d}\sigma^2 = \frac{\sum \mathrm{d}u_i^2}{(1 + (K/4)\sum u_i^2)^2}.$$

Replacing r by r/c and K by K/c^2, we obtain the following degeneration of Robertson–Walker metrics:

$$g = \mathrm{d}t^2 - \varepsilon r(t)^2 \frac{\sum \mathrm{d}u_i^2}{(1 + (\varepsilon K/4)\sum u_i^2)^2},$$

where $\varepsilon = c^{-2}$.

Since we have $g \equiv dt^2 \bmod (\varepsilon)$, the relative linear connection ∇ associated to g is prolongable (by Theorem 4.8). Computing the limit connection $\bar{\nabla}$ we obtain

$$\bar{\nabla}_{\partial_t}\partial_t = 0, \qquad \bar{\nabla}_{\partial_t}\partial_{u_i} = \frac{r'}{r}\partial_{u_i}, \qquad \bar{\nabla}_{\partial_{u_i}}\partial_{u_j} = 0.$$

Moreover, a direct computation gives us the following matrix for the relative curvature operator in the basis $\{\partial_t \wedge \partial_{u_1}, \partial_t \wedge \partial_{u_2}, \partial_t \wedge \partial_{u_3}, \partial_{u_2} \wedge \partial_{u_3}, \partial_{u_3}, \wedge\partial_{u_1}, \partial_{u_1} \wedge \partial_{u_2}\}$:

$$\mathcal{R} = \begin{pmatrix} -\dfrac{r''}{r}I & 0 \\ 0 & -\dfrac{K+(r')^2}{r^2}I \end{pmatrix},$$

where I is the 3×3 identity matrix. Therefore, $\mathcal{R}$ is prolongable and we conclude by Proposition 5.3 that the limit fibre $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla})$ is a Newtonian space–time.

The Ricci tensor of $\bar{\nabla}$ is $\bar{R}_2 = -3(r''/r)\,dt^2$. It follows that, in the limit Newtonian space–time, the mass density is

$$\rho = -\frac{3r''}{4\pi r}.$$

Remark that in this Newtonian space–time the coordinates (t, u_1, u_2, u_3) are *not* a Newtonian reference frame, because $\bar{g}^* = (1/r^2)\sum \partial_{u_i} \otimes \partial_{u_i}$. To get coordinates in a Newtonian reference frame it is enough to put $x_i = ru_i$. In this Newtonian reference frame (t, x_1, x_2, x_3) the intensity force is

$$F = -\bar{\nabla}_{\partial_t}\partial_t = \left(\frac{r''}{r}\right)(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}).$$

As it is well-known, a Robertson–Walker metric describes an isotropic cosmology. Its energy-impulse tensor $T^2 = (\rho + p)D \otimes D - pg^*$ corresponds to a relativistic perfect fluid, where the flow vector field is $D = \partial_t$ (written in the coordinates (t, u_1, u_2, u_3)), the energy density is $\rho = 3((r')^2 + K)/8\pi r^2$ and the pressure is $p = -(2r''r + (r')^2 + K)/8\pi r^2$ (see [15, 12.11]). It is immediate to check that the relative energy-impulse tensor T^2 of the degeneration is prolongable if and only if $p = 0$ (i.e., the fluid is a *dust*). In this case, the Newtonian limit $(X_0, \bar{g}, \bar{g}^*, \bar{\nabla}, \bar{T}^2)$ is a Newtonian perfect fluid with flow vector field $D = \partial_t + (r'/r)\sum x_i\partial_{x_i}$, mass density $\rho = -3r''/4\pi r$ and pressure $p = 0$. This Newtonian gravitation is the isotropic cosmology studied by Heckmann and Schucking [12].

7.3. Any Newtonian gravitation is the limit of a degeneration

A natural question is whether any Newtonian gravitation is the limit fibre of some suitable degeneration of Lorentz metrics, in such way that the Newtonian matter tensor $\bar{T}^2$ is the limit of the relativistic energy-impulse tensors. The answer is affirmative under certain conditions. Let us show, without proofs, the degeneration.

Let us consider a Newtonian gravitation on $\mathbb{R}^4$, where the Cartesian coordinates (t, x_1, x_2, x_3) are a Newtonian reference frame (Definition 2.3). Let u be the corresponding potential function. We shall assume that the matter tensor $\bar{T}^2$ has compact support at each instant t and that u is the Newtonian potential corresponding to the mass density ρ. Let us write

$$\bar{T}^2 = \rho \partial_t \otimes \partial_t + \sum w_i (\partial_t \otimes \partial_{x_i} + \partial_{x_i} \otimes \partial_t) + \sum h_{ij} \partial_{x_i} \otimes \partial_{x_j}.$$

Then the desired degeneration is

$$g = g_{tt}\, \mathrm{d}t \otimes \mathrm{d}t + \sum g_{ti} (\mathrm{d}t \otimes \mathrm{d}x_i + \mathrm{d}x_i \otimes \mathrm{d}t) + \sum g_{ij}\, \mathrm{d}x_i \otimes \mathrm{d}x_j,$$
$$g_{tt} = 1 - 2u\varepsilon + a\varepsilon^2 + \mathrm{O}(\varepsilon^3), \qquad g_{ti} = W_i \varepsilon^2 + \mathrm{O}(\varepsilon^3),$$
$$g_{ij} = -(\varepsilon + 2u\varepsilon^2)\delta_{ij} - H_{ij}\varepsilon^3 + \mathrm{O}(\varepsilon^4),$$

where W_i is the Newtonian potential corresponding to $4w_i$, H_{ij} the Newtonian potential corresponding to $4h_{ij} - (1/4\pi)(2(\sum u_k^2)\delta_{ij} + 4uu_{ij})$ and $a = 6u^2 - \sum H_{kk}$. (Notations: $u_k = \partial u / \partial x_k$, $u_{ij} = \partial^2 u / \partial x_i \partial x_j$.)

7.4. Newtonian gravitation defined by punctual masses

Let us consider n curves

$$\sigma_i : \mathbb{R} \to \mathbb{R}^4, \qquad \sigma_i(t) = (t, x_1^i(t), x_2^i(t), x_3^i(t)), \quad 1 \le i \le n.$$

Let us consider a Newtonian gravitation defined on the complement of the n curves in $\mathbb{R}^4$. We assume that the Cartesian coordinates t, x_1, x_2, x_3 are a Newtonian reference frame (2.3) with the following potential function:

$$u(t, x_1, x_2, x_3) = \sum_{i=1}^{n} \left(\frac{m_i}{r_i} \right), \quad r_i = \sqrt{\sum_k (x_k - x_k^i(t))^2},$$

where the terms m_i only depend on t, and r_i is the “spatial distance” to $\sigma_i(t)$. We also assume that the matter tensor $\bar{T}^2$ vanishes. This Newtonian gravitation will be said to be defined by n punctual masses (following the trajectories $\sigma_i(t)$ and with variable masses $m_i(t)$).

Note that the axioms of a Newtonian gravitation do not impose any condition to the trajectories $\sigma_i(t)$ neither to the masses $m_i(t)$. Although we prove in [13], using a global version of Theorem 6.5 (stated in Appendix B), the following theorem which is closely related to the classical EIH results [7,8].

Theorem 7.1. *If a Newtonian gravitation defined by n punctual masses is the limit fibre of a degeneration of Lorentz metrics then the masses $m_i(t)$ are constants and the trajectories $\sigma_i(t)$ obey the Newtonian law of motion.*

See Ehlers [6] for more examples of Newtonian limits. The best result on the existence of Newtonian limits is contained in Rendall [17].

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Appendix A. Relative differential forms

In this appendix we prove the relative versions of the Poincaré lemma and the Frobenius theorem used in this paper. The theory of differentiable spaces (see [14]) provides the natural context for this kind of results.

Lemma A.1. *Let L be a free module of finite rank over a local ring $\mathcal{O}$ and let P, P' be free submodules of L, both direct summands of rank r. If $\Lambda^r P = \Lambda^r P'$, then $P = P'$.*

Proof. Since $\mathcal{O}$ is a local ring, any direct summand of L is a free module. Therefore, we may choose a basis $\{e_1, \dots, e_n\}$ of L such that $\{e_1, \dots, e_r\}$ is a basis of P.

For any $e' \in P'$ we may write $e' = a_1 e_1 + \cdots + a_n e_n$ and

$$e_1 \wedge \cdots \wedge e_r \wedge e' = a_{r+1} e_1 \wedge \cdots \wedge e_r \wedge e_{r+1} + \cdots + a_n e_1 \wedge \cdots \wedge e_r \wedge e_n.$$

Since $e_1 \wedge \cdots \wedge e_r \wedge e' \in \Lambda^r P \wedge P' = \Lambda^{r+1} P' = 0$ we conclude that $a_{r+1} = \cdots = a_n = 0$, hence $e' \in P$. □

With the notations of the lemma, let I be an ideal of $\mathcal{O}$. Then P/IP and P'/IP' are free $\mathcal{O}/I$-modules and direct summands of L/IL. We write $P \equiv P' \bmod I$ when $P/IP = P'/IP'$. Since $\Lambda^r_{\mathcal{O}/I}(P/IP) = (\Lambda^r_{\mathcal{O}} P)/I\Lambda^r_{\mathcal{O}} P$ the above lemma implies the following.

Corollary A.2. *With the notations of the lemma, let I be an ideal of $\mathcal{O}$. If $\Lambda^r P \equiv \Lambda^r P' \bmod I$ then $P \equiv P' \bmod I$.*

Let us recall the notations of Section 3. Let $\varphi : X \to Y$ be a submersion. Given points $x \in X$ and $y = \varphi(x) \in Y$, let us consider the differential exterior algebra of the relative forms (germs at x)

$$\Omega^\bullet_x = \mathcal{C}^\infty_x \oplus \Omega^1_x \oplus \Omega^2_x \oplus \cdots .$$

Relative Frobenius Theorem A.3. *Let $P \subseteq \Omega^1_x$ be a relative Pfaff system at x and let $J \subseteq \mathcal{C}^\infty_y$ be an ideal. If $\mathrm{d}P \equiv 0 \bmod (J, P)$, i.e., $\mathrm{d}P = 0$ in the quotient algebra $\Omega^\bullet_x/(J, P)$, then P is integrable modulo (J),*

$$P \equiv \langle \mathrm{d}z_1, \dots, \mathrm{d}z_r \rangle \bmod (J).$$

Moreover, if P is also integrable modulo an ideal $J' \subseteq J$, then there exist germs $z'_i \equiv z_i \bmod (J)$ such that

$$P \equiv \langle \mathrm{d}z'_1, \dots, \mathrm{d}z'_r \rangle \bmod (J').$$

Proof. Let us consider local coordinates $(x_1, \dots, x_n, y_1, \dots)$ such that $\varphi(x_1, \dots, y_1, \dots) = (y_1, \dots)$. Then $\Omega^1_x = \langle \mathrm{d}x_1, \dots, \mathrm{d}x_n \rangle$. Let us write $P = \langle \omega_1, \dots, \omega_r \rangle$, where r is the rank of P. We prove the integrability of $P \bmod (J)$ by induction on the relative dimension n. The initial case $n = 1$ is obvious.

For the general case we may assume that $r < n$ since the case $r = n$ is obvious. Then there exists a relative vector field D such that $D_x \neq 0$ and $\omega_1(D) = \cdots = \omega_r(D) = 0$. Taking suitable coordinates we may assume that $D = \partial/\partial x_n$. For clarity let us write $t = x_n$, so that $D = \partial/\partial t$ and $\Omega^1_x = \langle \mathrm{d}x_1, \dots, \mathrm{d}x_{n-1}, \mathrm{d}t\rangle$. Let $\lambda = t(x)$. Let us consider the hypersurface X' of X defined by the equation $t = \lambda$. Then $\varphi : X \to Y$ is the composition of the following maps:

$$\pi : X \to X', \pi(x_1, \dots, t, y_1, \dots) = (x_1, \dots, \lambda, y_1, \dots), \quad \varphi' : X' \to Y, \varphi' = \varphi_{|X'}.$$

Note that $(x_1, \dots, x_{n-1}, y_1, \dots)$ are local coordinates on X' and that $\Omega^1_{X',x} = \langle \mathrm{d}x_1, \dots, \mathrm{d}x_{n-1}\rangle$, where $\Omega^1_{X',x}$ denotes the module of relative one-forms with respect to the map $\varphi' : X' \to Y$. Let $i : X' \hookrightarrow X$ be the natural inclusion. By the induction hypothesis the Pfaff system i^*P on X' is integrable $\bmod (J)$. Now we prove that $P \equiv \pi^* i^* P \bmod (J)$ and the integrability of $P \bmod (J)$ follows.

By hypothesis we have

$$\mathrm{d}\omega_j \equiv \eta_{1j} \wedge \omega_1 + \cdots + \eta_{rj} \wedge \omega_r \bmod (J)$$

for some $\eta_{ij} \in \Omega^1_x$. Therefore,

$$L_D \omega_j = i_D\, \mathrm{d}\omega_j \equiv \sum_i \eta_{ij}(D)\omega_i \bmod (J),$$

hence

$$L_D(\omega_1 \wedge \cdots \wedge \omega_r) \equiv u\omega_1 \wedge \cdots \wedge \omega_r \bmod (J)$$

for certain smooth function u. Replacing ω_1 by $\mathrm{e}^{-\int u\, \mathrm{d}t}\omega_1$ we may assume that

$$L_D(\omega_1 \wedge \cdots \wedge \omega_r) \equiv 0 \bmod (J).$$

Since $\omega_j(\partial/\partial t) = 0$ we have

$$\omega_1 \wedge \cdots \wedge \omega_r = \sum_{\alpha=(i_1<\cdots<i_r)} g_\alpha\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r}$$

for certain smooth functions g_α. Then

$$0 \equiv L_D(\omega_1 \wedge \cdots \wedge \omega_r) = \sum_{\alpha=(i_1<\cdots<i_r)} \frac{\partial g_\alpha}{\partial t} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r} \bmod (J),$$

hence $\partial g_\alpha/\partial t \equiv 0 \bmod (J)$, i.e., $\partial g_\alpha/\partial t = \sum f_i h_i$ for certain $h_i \in J$. Then

$$g_\alpha = \bar{g}_\alpha(x_1, \dots, x_{n-1}, y_1, \dots) + \sum h_i \int f_i\, \mathrm{d}t \equiv \bar{g}_\alpha(x_1, \dots, x_{n-1}, y_1, \dots) \bmod (J),$$

where $\bar{g}_\alpha$ does not depend on t. As a consequence it results that $\pi^* i^*(g_\alpha) \equiv \pi^* i^*(\bar{g}_\alpha) = \bar{g}_\alpha \equiv g_\alpha \bmod (J)$ and therefore

$$\pi^* i^*(\omega_1 \wedge \cdots \wedge \omega_r) \equiv \omega_1 \wedge \cdots \wedge \omega_r \bmod (J).$$

From Corollary A.2 we conclude that $\pi^* i^* P \equiv P \bmod (J)$.

Now we prove the second part of the statement by induction on the relative dimension n. It is assumed that P is integrable $\bmod (J)$, i.e., $P \equiv \langle \mathrm{d}z_1, \dots, \mathrm{d}z_r \rangle \bmod (J)$ and that P also is integrable modulo an ideal $J' \subset J$.

With the previous notations we have that $0 \equiv \mathrm{d}z_i(D) = \partial z_i / \partial t \bmod (J)$. The same argument used for g_α proves that $\pi^* i^*(z_j) \equiv z_j \bmod (J)$.

Restricting to the hypersurface X' we have $i^* P \equiv \langle \mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_r \rangle \bmod (J)$, where $\bar{z}_j := i^* z_j$. By induction hypothesis there exist smooth functions $\bar{z}'_j \equiv \bar{z}_j \bmod (J)$ such that $i^* P \equiv \langle \mathrm{d}\bar{z}'_1, \dots, \mathrm{d}\bar{z}'_r \rangle \bmod (J')$. Therefore,

$$P \equiv \pi^* i^* P \equiv \langle \mathrm{d}z', \dots, \mathrm{d}z'_r \rangle \bmod (J),$$

where $z'_j := \pi^*(\bar{z}'_j) \equiv \pi^*(\bar{z}_j) = \pi^* i^* z_j \equiv z_j \bmod (J)$. □

In this paper, we have used the Frobenius theorem in the following slightly more general version.

Relative Frobenius Theorem A.3′. *Let $J \subseteq \mathcal{C}^\infty_y$ be an ideal and let $P = M \oplus N \subseteq \Omega^1_x$ be a relative Pfaff system at x such that N is integrable modulo (J). If $\mathrm{d}M \equiv 0 \bmod (J, P)$ then M is integrable $\bmod (J, N)$, i.e.,*

$$M \equiv \langle \mathrm{d}z_1, \dots, \mathrm{d}z_r \rangle \quad \text{in} \quad \Omega^\bullet_x/(J, N).$$

Proof. By Relative Frobenius Theorem A.3 the hypotheses imply that P is integrable $\bmod (J)$, i.e.,

$$P/JP = \langle \mathrm{d}z_1, \dots, \mathrm{d}z_m \rangle \subseteq \Omega^1_x / J\Omega^1_x.$$

We conclude the proof taking quotient with respect to N

$$M/JM = \langle \mathrm{d}z_1, \dots, \mathrm{d}z_m \rangle \subseteq \Omega^1_x / (J\Omega^1_x + N).$$ □

Proposition A.4. *Let $\varepsilon : X \to \mathbb{R}$ be a submersion and let $\langle \theta, \omega_1, \dots, \omega_k \rangle$ be a relative Pfaff system (at a point $x \in X$ with $\varepsilon(x) = 0$) of rank $k + 1$. If θ is exact $\bmod (\varepsilon^r)$ and*

$$\mathrm{d}\omega_i \equiv 0 \bmod (\varepsilon^{2r}, \varepsilon^r \theta, \omega_1, \dots, \omega_k), \qquad 1 \le i \le k$$

then $\langle \omega_1, \dots, \omega_k \rangle$ is integrable $\bmod (\varepsilon^{2r}, \varepsilon^r \theta)$.

Proof. Since θ is exact $\bmod (\varepsilon^r)$ we may put

$$\theta = \mathrm{d}t + \varepsilon^r \theta'.$$

By the Frobenius theorem $\langle \omega_1, \dots, \omega_k \rangle$ is integrable $\bmod (\varepsilon^r)$

$$\langle \omega_1, \dots, \omega_k \rangle \equiv \langle \mathrm{d}x_1, \dots, \mathrm{d}x_k \rangle \bmod (\varepsilon^r).$$

Replacing the generators ω_i if necessary, we may assume that

$$(*) \quad \omega_i = \mathrm{d}x_i + \varepsilon^r \omega'_i.$$

Since $\varepsilon^r\theta \equiv \varepsilon^r\, \mathrm{d}t \bmod (\varepsilon^{2r})$, the conditions $\mathrm{d}\omega_i \equiv 0 \bmod (\varepsilon^{2r}, \varepsilon^r\theta, \omega_1, \dots, \omega_k)$ imply that the Pfaff system $\langle \mathrm{d}t, \omega_1, \dots, \omega_k\rangle$ also is integrable $\bmod (\varepsilon^{2r})$, i.e.,

$$\langle \mathrm{d}t, \omega_1, \dots, \omega_k\rangle \equiv \langle \mathrm{d}t', \mathrm{d}x'_1, \dots, \mathrm{d}x'_k\rangle \bmod (\varepsilon^{2r}).$$

Using the second part of Relative Frobenius Theorem A.3 we may put $t' = t + \varepsilon^r v$ and $x'_i = x_i + \varepsilon^r u_i$. Hence, $\mathrm{d}x'_i = \mathrm{d}x_i + \varepsilon^r\, \mathrm{d}u_i$, and using (∗) it follows easily that

$$\langle \varepsilon^r\, \mathrm{d}t, \omega_1, \dots, \omega_k\rangle \equiv \langle \varepsilon^r\, \mathrm{d}t, \mathrm{d}x'_1, \dots, \mathrm{d}x'_k\rangle \bmod (\varepsilon^{2r}),$$

and therefore

$$\langle \omega_1, \dots, \omega_k\rangle \equiv \langle \mathrm{d}x'_1, \dots, \mathrm{d}x'_k\rangle \bmod (\varepsilon^{2r}, \varepsilon^r\, \mathrm{d}t) = (\varepsilon^{2r}, \varepsilon^r\theta). \qquad \square$$

Let us again consider a submersion $\varphi : X \to Y$ and respective points $x \in X$ and $y = \varphi(x) \in Y$. Let J be an ideal of $\mathcal{C}^\infty_y$.

Relative Poincaré Lemma A.5. *A relative p-form $\omega \in \Omega^p_x$ is closed modulo (J) if and only if it is exact modulo (J).*

Proof. Of course any exact $\bmod (J)$ p-form is closed $\bmod (J)$. For the inverse let us consider local coordinates $(x_1, \dots, x_n, y_1, \dots)$ such that $\varphi(x_1, \dots, y_1, \dots) = (y_1, \dots)$. Let us also consider the relative vector field $D := x_1(\partial/\partial x_1) + \cdots + x_n(\partial/\partial x_n)$ and the operator $H : \Omega^\bullet_x \to \Omega^\bullet_x$ defined by the equalities

$$\omega = \sum_{j_1<\cdots<j_p} f_{j_1\cdots j_p}(x_1, \dots, x_n, y_1, \dots)\, \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p},$$

$$H\omega = \sum_{j_1<\cdots<j_p} \left(\int_0^1 t^{p-1} f_{j_1\cdots j_p}(tx_1, \dots, tx_n, y_1, \dots)\, \mathrm{d}t \right) i_D(\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_p}).$$

The standard proof of the absolute Poincaré lemma shows that $\mathrm{Id} = H \circ \mathrm{d} + \mathrm{d} \circ H$. To conclude it is enough to remark that $H\omega \in J\Omega^{p-1}_x$ whenever $\omega \in J\Omega^p_x$. If $f \in (J) = J\mathcal{C}^\infty_x$, i.e.,

$$f(x_1, \dots, x_n, y_1, \dots) = \sum_i h_i(y_1, \dots) g_i(x_1, \dots, x_n, y_1, \dots)$$

for certain $h_i \in J$, then

$$\int_0^1 t^{p-1} f(tx_1, \dots, tx_n, y_1, \dots)\, \mathrm{d}t = \sum_i h_i(y_1, \dots) \int_0^1 t^{p-1} g_i(tx_1, \dots, tx_n, y_1, \dots)\, \mathrm{d}t \in (J). \qquad \square$$

Let us consider a more general version of the Poincaré lemma.

Relative Poincaré Lemma A.5′. *Let N be an integrable* $(\bmod J)$ *Pfaff system at x and let* $\omega \in \Omega_x^p$ *be a relative p-form. If* $\mathrm{d}\omega \equiv 0 \bmod (J, N)$ *then* $\omega \equiv \mathrm{d}\omega' \bmod (J, N)$ *for some relative* $(p-1)$*-form* $\omega' \in \Omega_x^{p-1}$.

Proof. Since N is integrable $\bmod (J)$ we may write $N \equiv \langle \mathrm{d}t_1, \dots, \mathrm{d}t_r\rangle \bmod (J)$, where r is the rank of N. Then $(J, N) = (J, \mathrm{d}t_1, \dots, \mathrm{d}t_n)$ (ideals of $\Omega_x^\bullet$). This implies that

$$\Omega_x^\bullet/(J, N) = \tilde{\Omega}_x^\bullet/(J),$$

where $\tilde{\Omega}_x^\bullet$ is the algebra of relative forms with respect to the submersion $(\varphi, t_1, \dots, t_r) : X \to Y \times \mathbb{R}^r$.

Now the result follows from Relative Poincaré Lemma A.5 applied to $\tilde{\Omega}_x^\bullet/(J)$. □

Proposition A.6. *Let* $\varepsilon : X \to \mathbb{R}$ *be a submersion and let* θ, ω *be relative* 1*-forms (germs at a point* $x \in X$ *with* $\varepsilon(x) = 0$*). Let us assume that* $\theta_x \neq 0$*, i.e.,* $\langle\theta\rangle$ *is a Pfaff system. If* θ *is exact modulo* (ε^r) *and*

$$\mathrm{d}\omega \equiv 0 \bmod (\varepsilon^{2r}, \varepsilon^r\theta)$$

then ω *is exact* $\bmod (\varepsilon^{2r}, \varepsilon^r\theta)$.

Proof. Note that $\mathrm{d}\omega \equiv 0 \bmod (\varepsilon^r)$. By the Poincaré lemma we may write

$$\omega = \mathrm{d}x + \varepsilon^r\omega'$$

for some relative one-form ω'. By hypothesis, $0 \equiv \mathrm{d}\omega = \varepsilon^r\,\mathrm{d}\omega' \bmod (\varepsilon^{2r}, \varepsilon^r\theta)$. Hence, $\mathrm{d}\omega' \equiv 0 \bmod (\varepsilon^r, \theta)$. By the Relative Poincaré Lemma A.5′, it results in $\omega' \equiv \mathrm{d}f \bmod (\varepsilon^r, \theta)$ for certain function f. Therefore,

$$\omega = \mathrm{d}x + \varepsilon^r\omega' \equiv \mathrm{d}x + \varepsilon^r\,\mathrm{d}f \equiv \mathrm{d}(x + \varepsilon^r f) \bmod (\varepsilon^{2r}, \varepsilon^r\theta).$$

□

Appendix B. A global result

In this appendix, we shall obtain a global version of Theorem 6.5. Let us begin with some observations on Newtonian reference frames (Definition 2.3):

(a) The Newtonian potential corresponding to a Newtonian reference frame (t, x_1, x_2, x_3) is uniquely determined except by a summand $h(t)$.
(b) Two Newtonian reference frames are related by a transformation of the type

$$\bar{t} = a + t, \qquad \bar{x}_i = f_i(t) + \sum_j a_{ij}x_j,$$

where a is a constant and (a_{ij}) is an orthogonal matrix of constant coefficients.

Let (X, ε, g) be a degeneration of Lorentz metrics such that the relative connection ∇, the curvature operator $\mathcal{R}$ and the Einstein tensor G^2 are prolongable. By Theorem 6.2, the limit fibre X_0 inherits the structure of a Newtonian gravitation.

Lemma B.1. *Let $(\bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ be a local Newtonian reference frame on X_0 and let $\bar{u}$ be the corresponding potential function. These functions may be extended locally so as to obtain a local coordinate system $(\varepsilon, \bar{t}, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ on X, such that the coefficients of g in that system are*

$$\bar{g}_{tt} \equiv 1 - 2\bar{u}\varepsilon \bmod (\varepsilon^2), \qquad \bar{g}_{ti} \equiv 0 \bmod (\varepsilon^2), \qquad \bar{g}_{ij} \equiv -(\varepsilon + 2\bar{u}\varepsilon^2)\delta_{ij} \bmod (\varepsilon^3).$$

Proof. By Theorem 6.5 there exists a local coordinate system $(\varepsilon, t, x_1, x_2, x_3)$ on X such that the corresponding coefficients of g are

$$g_{tt} \equiv 1 - 2u\varepsilon \bmod (\varepsilon^2), \qquad g_{ti} \equiv 0 \bmod (\varepsilon^2), \qquad g_{ij} \equiv -(\varepsilon + 2u\varepsilon^2)\delta_{ij} \bmod (\varepsilon^3).$$

Moreover, the restriction of (t, x_1, x_2, x_3) on X_0 is a Newtonian reference frame, u being the corresponding potential function (see Remark 6.7). Recalling the above observations (a) and (b), it is easy to check that the desired coordinate system is obtained by means of a suitable transformation

$$\bar{t} = a + t + \varepsilon h(t), \qquad \bar{x}_i = f_i(t) + \sum_j a_{ij} x_j. \qquad \square$$

Theorem B.2. *Let (t, x_1, x_2, x_3) be a global Newtonian reference frame on X_0 such that the slices $t =$ const. are simply connected. This reference may be extended so as to obtain a coordinate system $(\varepsilon, t, x_1, x_2, x_3)$ on a neighborhood of X_0, such that the corresponding coefficients of g are*

$$g_{tt} \equiv 1 - 2u\varepsilon \bmod (\varepsilon^2), \qquad g_{ti} \equiv 0 \bmod (\varepsilon^2), \qquad g_{ij} \equiv -(\varepsilon + 2u\varepsilon^2)\delta_{ij} \bmod (\varepsilon^3).$$

Proof. The previous lemma says that the desired extension locally exists. This local extension is not unique. It is easy to check that two extensions are related by a transformation of the type

$$\bar{t} \equiv t + \varepsilon a \bmod (\varepsilon^2), \qquad \bar{x}_i \equiv x_i + \varepsilon \left(h_i(t) + \sum_j h_{ij}(t) x_j \right) \bmod (\varepsilon^2),$$

where a is locally constant, $h_i(t)$ are locally smooth functions on t, and $h_{ij}(t)$ are locally smooth functions on t such that $h_{ij}(t) = -h_{ji}(t)$. Therefore, the obstruction to the existence of a global extension yields in the cohomology group $H^1(X_0, \mathbb{R} \oplus \mathcal{O}^6) = H^1(X_0, \mathbb{R}) \oplus H^1(X_0, \mathcal{O})^6$, where $\mathbb{R}$ is the sheaf of locally constant functions and $\mathcal{O}$ is the sheaf of smooth functions on t (locally). Since the slices $t =$ const. are simply connected, we have that X_0 is also simply connected and then $H^1(X_0, \mathbb{R}) = 0$.

Finally we have to show that $H^1(X, \mathcal{O}) = 0$. Let $\Omega^p_{X_0/\mathbb{R}}$ be the sheaf on X_0 of relative p-forms with respect to the submersion $t : X_0 \to \mathbb{R}$. Let us consider the relative De Rham complex of sheaves

$$0 \to \mathcal{O} \to \mathcal{C}^\infty_{X_0} \xrightarrow{d} \Omega^1_{X_0/\mathbb{R}} \xrightarrow{d} \Omega^2_{X_0/\mathbb{R}} \xrightarrow{d} \Omega^3_{X_0/\mathbb{R}} \to 0.$$

This sequence is exact by the Relative Poincaré Lemma A.5. The sheaves $\Omega^p_{X_0/\mathbb{R}}$ are $\mathcal{C}^\infty_{X_0}$-modules hence they are soft (see [11]) and the above sequence is an acyclic resolution of $\mathcal{O}$. Therefore,

$$H^1(X_0, \mathcal{O}) = \left\{ \frac{\text{relative closed 1-forms on } X_0}{\text{relative exact 1-forms on } X_0} \right\},$$

and we have to prove that any relative closed one-form ω is exact. Let $\sigma : \mathbb{R} \to X_0$ be a smooth section of $t : X_0 \to \mathbb{R}$. For any point $(t, x) \in X_0$ we define

$$F(t, x) = \int_\gamma \omega,$$

where γ is an arc joining $\sigma(t)$ and (t, x) contained in a slice $t = \text{const}$. Since the slice is simply connected, the above integral does not depend on the chosen arc and we have $\omega = \mathrm{d}F$. □

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